The Effects of Vertical Deflections on Aircraft Inertial Navigation Systems

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Autocorrelation functions (ACFs) of surface gravity anomalies are used as a method of predicting expected aircraft navigation errors induced by deflection model uncertainties. Theoretical methods are developed to propagate ACFs from surface to aircraft altitudes, to determine ACFs of vertical deflection components from ACFs for gravity anomalies, and to predict average position and velocity uncertainties in the aircraft navigation system that arise from uncertainties in modeling vertical deflections. Numerical results include a study of navigation error sensitivities to system parameters and estimates of representative navigation errors implied by using several specific deflection models in the United States.

Introduction

EXTREMELY accurate navigation for a cruising aircraft can only be achieved by a combination of several navigation subsystems. These subsystems should include an inertial navigation system, a velocity reference such as a Doppler radar, and a position reference such as an altimeter or terrain radar. As discussed in Refs. 1 and 2, important contributions to the horizontal navigation errors in such a system arise from uncertainties in components of deflections of the vertical.

The approach of this paper is to use ACFs and power spectral densities (PSDs) of surface gravity anomalies (Δgs) as a starting point from which to predict expected aircraft navigation errors. Three conceptual steps are required in this process:

1) Compute the ACF or PSD of Δg at cruise altitude h given the ACF or PSD at the surface.

2) Compute the ACF or PSD of deflection components (ξ and η) at altitude h from the output of step 1.

3) Compute the navigation error statistics at altitude from the output of step 2. Steps 2 and 3 are discussed for surface cruise vehicles in Refs. 1-3, and step 1 is discussed in Refs. 4 and 5 for isotropic ACFs. This paper expands and combines the theoretical treatments of these references to develop numerical results for a typical set of simple navigation equations. These results include navigation error sensitivities to certain system parameters and estimates of typical navigation errors implied by the use of several specific gravity models in the United States.

Continuation of the ACF for Δg from the Surface to Altitude

The general upward continuation formulas for the PSD, S, and the ACF, R, are

$$S(k,h)=S(k,0)e^{-2kh} \tag{1}$$

$$R(\tau,h)=\frac{h}{\pi}\int \mathrm{d}\tau' \frac{R(\tau',0)}{[\,|\tau-\tau'|^2+4h^2]^{3/2}} \tag{2}$$

where τ is shift distance and k is wave number. The derivation of Eqs. (1) and (2) is given in Appendix A and relies on two simplifications: the Earth is flat, and Δg is harmonic. A similar derivation is given by Bellaire;[4] he also discusses the errors made by the flat-Earth assumptions in Ref. 5 and concludes that the errors are substantially less than 1% for altitudes under 10 km. The simplification that Δg is harmonic follows from the flat-Earth simplification and the fact that $r\Delta g$ is a harmonic function (for example see Ref. 6).

When $R(\tau,h)$ is assumed to be independent of the direction of τ (isotropic), the angular integral of Eq. (2) can be evaluated analytically to give

$$R(\tau,h)=\frac{4h}{\pi}\int_0^\infty \frac{R(\tau',0)E(P)\tau' d\tau'}{[(\tau-\tau')^2+4h^2][(\tau+\tau')^2+4h^2]^{1/2}} \tag{3}$$

where $E(P)$ is the complete elliptical integral of the second kind and

$$P^2=\frac{4\tau\tau'}{(\tau+\tau')^2+4h^2} \tag{4}$$

A computer program was written to evaluate the integral in Eq. (3) numerically and results were generated for two example ACFs:

1) The simple exponential

$$R_1(\tau,0)=e^{-\tau/d} \tag{5}$$

2) The linear exponential

$$R_2(\tau,0)=(1+2.15\tau/d)e^{-2.15\tau/d} \tag{6}$$

Numerical results for a correlation distance $d=40$ nmi and altitudes of 20 kft and 40 kft are shown in Fig. 1. The effect of altitude is to reduce the ACF for relatively short shift distances but to leave it essentially unchanged for the longer shift distances.

Transformation of ACFs for Δg to ACFs for Deflection Components

The general relations between ACFs and PSDs for Δg and gravity deflection components are

$$S_{\left\{\substack{\eta\\ \xi}\right\}}(k)=\frac{1}{g^2}\left\{\begin{matrix}\sin^2\theta_k\\ \cos^2\theta_k\end{matrix}\right\}S_{\Delta g}(k) \tag{7}$$

where θ_k is the direction of k measured from the north, g is the magnitude of gravity, and η and ξ are the east and north vertical deflections.

The ACFs of η and ξ are given by

$$R_{\left\{\substack{\eta\\ \xi}\right\}}(\tau)=\frac{1}{g^2}\int dk \left\{\begin{matrix}\sin^2\theta_k\\ \cos^2\theta_k\end{matrix}\right\}e^{ik\cdot\tau}S_{\Delta g}(k) \tag{8}$$

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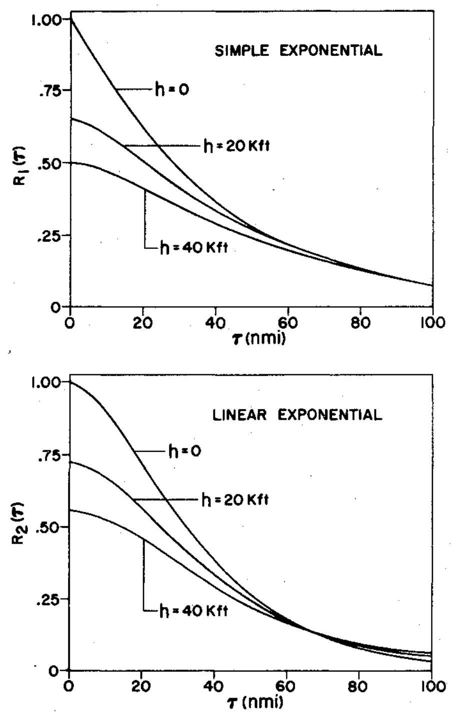

Fig. 1 ACFs at altitude for a 40-nmi correlation distance.

The derivations of Eqs. (7) and (8) begin with the Stokes integral, which expresses ξ and η as an integral of Δg times the Stokes function, S. Two simplifications are made in deriving Eqs. (7) and (8): 1) the Earth is made flat and the integration limits are extended to ∞; 2) the Stokes function, S, is replaced by the term that dominates at points near the origin of coordinates.

When $R_{\Delta g}(\tau)$ is assumed to be isotropic, the angular integral of Eq. (8) can be evaluated analytically to give

$$R_{\left\{ \begin{smallmatrix} \eta \\ \xi \end{smallmatrix} \right\}}(\tau,\theta) = \frac{1}{2g^2}\left[R_{\Delta g}(\tau) \pm \cos 2\theta f_c(\tau)\right] \tag{9}$$

where θ is the shift direction relative to north and $f_c(\tau)$ has the general form

$$f_c(\tau) = 2\pi \int_0^\infty k J_2(k\tau) S_{\Delta g}(k) dk \tag{10}$$

$J_2(k\tau)$ is the Bessel function of order 2. Numerical computation of Eq. (9) for the simple and linear exponential ACFs shows that $R_{\Delta g}$ dominates for small shift distances $(\tau < 0.5d)$ and that f_c dominates for large shift distances $(\tau > 3d)$.

Navigation Error Statistics

The linearized equations for horizontal (east and north) position errors are

$$\delta \ddot{r}_E + 2\zeta\omega_s \delta \dot{r}_E + \omega_s^2 \delta r_E - 2\Omega_Z \delta \dot{r}_N = -g\delta\eta \tag{11}$$

$$\delta \ddot{r}_N + 2\zeta\omega_s \delta \dot{r}_N + \omega_s^2 \delta r_N + 2\Omega_Z \delta \dot{r}_E = -g\delta\xi \tag{12}$$

where δr_E, δr_N are the east and north position errors, Ω_Z is the vertical component of the earth's rotation frequency, $\omega_s = (g/R)^{1/2}$ is the Schuler frequency, ζ is a velocity damping parameter, and $\delta\eta$ and $\delta\xi$ are deflection errors. Since this study isolates vertical deflection errors, a simple deflection compensation scheme is assumed in Eqs. (11) and (12). As discussed in Ref. 7, a more complete navigation system analysis would also include models for the Schuler loops as well as other error sources, and would use a Kalman filter to estimate position and velocity.

Equations (11) and (12) do not include third-order damping, which would probably be part of a real system in order to remove the effects of a bias in the Doppler information. The case of third-order damping is discussed in Appendix B, and its effects are included in the transfer functions used in the next section to compute numerical results. These results show only a slight sensitivity of navigation errors to the effects of third-order damping.

The solutions to Eqs. (11) and (12) show that horizontal position errors tend to oscillate with a frequency near ω_s, corresponding to a period of 86 min. The use of Doppler information serves to damp out these oscillations, with the damping depending on the damping parameter ζ. In the above equation, the velocity reference is assumed to be perfect. The Earth rotation couples the east and north error components, but because $\Omega_Z^2 \ll \omega_s^2$, the effect is very small.

If the coupling effects are ignored, the PSDs for east and north position errors can be related to the PSDs of deflection component errors by

$$S_{\left\{ \begin{smallmatrix} E \\ N \end{smallmatrix} \right\}}(\omega) = g^2 |H(\omega)|^2 S_{\left\{ \begin{smallmatrix} \eta \\ \xi \end{smallmatrix} \right\}}(\omega) \tag{13}$$

where the square of the system transfer function is

$$|H(\omega)|^2 = 1/[(\omega^2 - \omega_s^2)^2 + 4\zeta^2\omega_s^2\omega^2] \tag{14}$$

Equations (13), (7), and (1) can now be combined and the inverse Fourier transform taken to yield ACFs for east and north position errors

$$R_{\left\{ \begin{smallmatrix} E \\ N \end{smallmatrix} \right\}}(t) = \int d\omega e^{i\omega \cdot t} |H(\omega)|^2 \left\{ \begin{matrix} \sin^2\theta_\omega \\ \cos^2\theta_\omega \end{matrix} \right\} e^{-2\omega h/V} S_{\Delta g}(\omega) \tag{15}$$

In Eq. (15), the time and frequency domains have been used which are related simply to the τ and k domains by the vehicle speed, V.

$$t = \tau / V \tag{16}$$

$$\omega = kV \tag{17}$$

The variances in navigation errors are given by Eq. 15 when $t=0$. For this case the dependence on the direction of t disappears. If it is further assumed that $R_{\Delta g}$ is isotropic, then the variances reduce to particularly simple forms

$$\sigma_{r_E}^2 = \sigma_{r_N}^2 = \sigma_r^2 = F_1 \tag{18}$$

$$\sigma_{\dot{r}_E}^2 = \sigma_{\dot{r}_N}^2 = \sigma_{\dot{r}}^2 = F_3 \tag{19}$$

$$\sigma_{r_E \dot{r}_E}^2 = \sigma_{r_N \dot{r}_N}^2 = F_2 \tag{20}$$

where:

$$F_n = \pi \int_0^\infty \omega^n |H(\omega)|^2 e^{-2\omega h/V} S_{\Delta g}(\omega) d\omega \tag{21}$$

All cross-correlations between east and north errors are zero. If the form of $|H(\omega)|^2$ is appropriately changed, Eq. (21) is also valid for the coupled case where $\Omega_Z \neq 0$ and for the more realistic navigation equations, which include third-order damping.

Numerical Results

Equations (18) through (21), including earth rotation, were used to determine the sensitivities of navigation error components to each of the following system parameters: h = aircraft cruise altitude, V = aircraft cruise velocity, d = correlation distance, ζ = damping parameter, K_2 = third-order damping constant.

Figures 2-6 show these sensitivities for an input uncertainty in Δg of $\sigma_g = 1$ mgal ($=0.001$ cm/sec^2) for the simple and linear exponential models. Nominal values for each of the parameters are $h = 20$ kft, $V = 500$ knots, $d = 40$ nmi, $\zeta = 0.1$, $K_2 = 0.000285$ sec^{-1}, and $\Omega_Z = \Omega \sin 45°$. The effects of vehicle velocity and correlation distance are determined primarily by the ratio V/d, with maximum errors occurring when $V/d \approx 3$ hr^{-1}. Navigation errors are relatively insensitive to cruise altitude and third-order damping constant. Navigation errors are sensitive to the damping parameter ζ, with errors decreasing as ζ increases. However, choosing a large value of ζ is only allowable if the Doppler velocity reference system is sufficiently accurate. A study of the sensitivity of navigation errors to Ω_Z showed that including Ω_Z changed navigation errors by less than 1 part in 200.

Navigation errors presented in Figs. 2-6 are for a deflection compensation model that uses the Stokes integrals to compute deflection components from Δgs having an RMS uncertainty normalized to 1 mgal. In order to study the effects of more realistic models, the ACFs of Ref. 8 were input to Eqs. (18-21). These ACFs were developed from Δg models formed by averaging detailed 5 nmi × 5 nmi Δg data over increasingly larger areas and using the 5 nmi × 5 nmi data as a truth model. The resulting ACFs are averages of many ACFs for Δg taken along tracks in the United States and fit to the simple exponential form. Table 1 shows the position error, velocity error, and correlation between the two at three altitudes for each of six gravity models. The "worst case" and "best case" ACFs refer to an average of the respective 5% of the tracks having largest and smallest σ_g for their ACFs. They therefore reflect the variability that might be encountered as geographic position changes.

Not only are the navigation errors substantial for gravity models that are grosser than 300 nmi × 300 nmi, but they are also appreciable for even the 15 nmi × 15 nmi model. Also note that the variability between the "best case" and "worst case" is substantial, so that large fluctuations in navigation performance can be anticipated depending upon which track is taken.

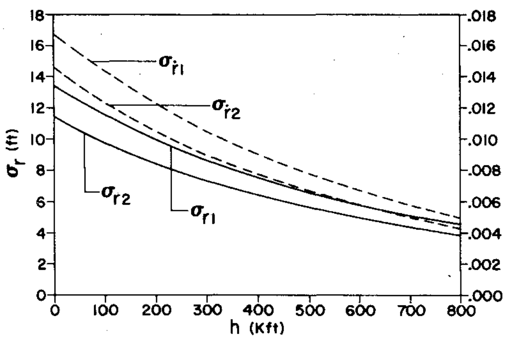

Fig. 2 Effect of cruise altitude on navigation errors.

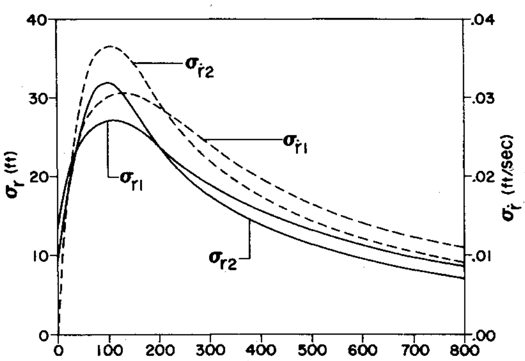

Fig. 3 Effect of cruise velocity on navigation errors.

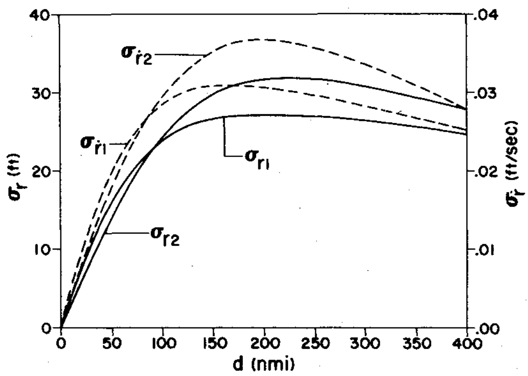

Fig. 4 Effect of correlation distance on navigation errors.

Appendix A. Upward Continuation of the ACF of a Harmonic Function

Assume that a harmonic function, f, is given on an infinite plane (such as the flat Earth). The two-dimensional ACF of the function can then be calculated

$$R(\tau_1,\tau_2) = \lim_{T\to\infty} \frac{1}{4T^2} \int_{-T}^{T} dx \int_{-T}^{T} dy\, f(x,y) f(x+\tau_1, y+\tau_2) \tag{A1}$$

If the Fourier transform of f is defined, the ACF can be expressed in terms of the PSD

$$\hat{f}(k_1,k_2) = \lim_{T\to\infty} \frac{1}{4T^2} \frac{1}{2\pi} \int_{-T}^{T} dx \int_{-T}^{T} dy\, f(x,y) e^{-i(k_1 x + k_2 y)} \tag{A2}$$

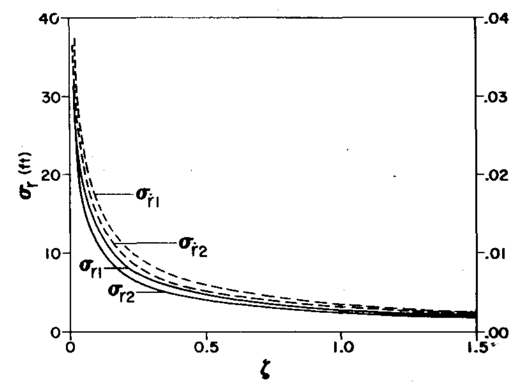

Fig. 5 Effect of damping parameter on navigation errors.

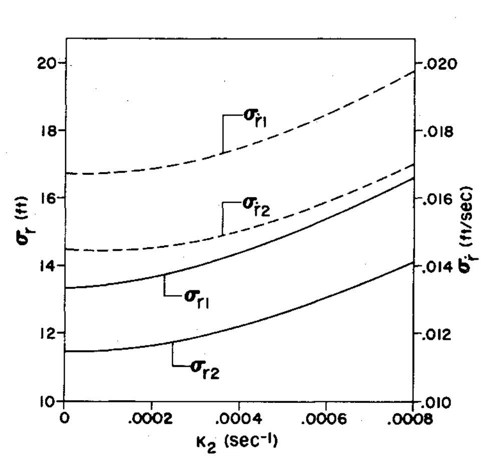

Fig. 6 Effect of third-order damping constant on navigation errors.

Table 1 Navigation errors for six gravity models fit to the simple exponential ACF ($V=500$ knots, $\zeta=0.1$, $K_2=0.000285\ \mathrm{sec}^{-1}$, $\Omega_Z=\Omega\sin 45°$)

		"Worst Case" ACF					Average ACF					"Best Case" ACF				
Gravity Model for Δg	h (Kft)	σ_g^2 (mgal^2)	d (nmi)	σ_r (ft)	$\sigma_{\dot r}$ (ft/sec)	Correlation	σ_g^2 (mgal^2)	d (nmi)	σ_r (ft)	$\sigma_{\dot r}$ (ft/sec)	Correlation	σ_g^2 (mgal^2)	d (nmi)	σ_r (ft)	$\sigma_{\dot r}$ (ft/sec)	Correlation
Ellipsoidal (Δg = 0)	0	3260	400	1482	1.508	.93	760	100	685	.805	.97	180	40	184	.226	.98
	20	3260	400	1450	1.467	.93	760	100	667	.782	.97	180	40	179	.219	.98
	40	3260	400	1418	1.427	.92	760	100	649	.759	.97	180	40	174	.212	.98
600 nmi × 600 nmi, averaged Δg's	0	2500	80	1119	1.333	.98	625	50	410	.500	.98	200	30	151	.186	.98
	20	2500	80	1089	1.294	.98	625	50	399	.485	.98	200	30	147	.181	.98
	40	2500	80	1059	1.255	.97	625	50	388	.470	.98	200	30	143	.175	.98
300 nmi × 300 nmi, averaged Δg's	0	2200	50	770	.938	.98	550	40	322	.395	.98	160	35	155	.191	.98
	20	2200	50	749	.909	.98	550	40	313	.383	.98	160	35	151	.185	.98
	40	2200	50	728	.882	.98	550	40	304	.371	.98	160	35	146	.179	.98
60 nmi × 60 nmi, averaged Δg's	0	1600	25	362	.448	.98	360	25	172	.212	.98	60	20	57	.070	.98
	20	1600	25	351	.434	.98	360	25	167	.206	.98	60	20	55	.068	.98
	40	1600	25	342	.420	.98	360	25	162	.199	.98	60	20	54	.066	.98
30 nmi × 30 nmi, averaged Δg's	0	1300	25	326	.403	.98	250	20	116	.144	.98	30	15	30	.038	.98
	20	1300	25	317	.391	.98	250	20	113	.139	.98	30	15	30	.037	.98
	40	1300	25	308	.379	.98	250	20	109	.134	.98	30	15	29	.035	.98
15 nmi × 15 nmi, averaged Δg's	0	637	30	270	.333	.98	115	20	79	.097	.98	15	10	14	.018	.98
	20	637	30	262	.323	.98	115	20	76	.094	.98	15	10	14	.017	.98
	40	637	30	255	.313	.98	115	20	74	.092	.98	15	10	14	.017	.98

$$S(k_1,k_2)=|\hat f(k_1,k_2)|^2$$

$$R(\tau_1,\tau_2)=\int_{-\infty}^{\infty}dk_1\int_{-\infty}^{\infty}dk_2S(k_1,k_2)e^{i(k_1\tau_1+k_2\tau_2)}\qquad\text{(A3)}$$

If the two-dimensional vector notation $\tau=(\tau_1,\tau_2)$ and $k=(k_1,k_2)$ is used, Eq. (A3) is more neatly written

$$R(\tau)=\int dkS(k)e^{ik\cdot\tau}\qquad\text{(A4)}$$

The object is to find the dependence of $R(\tau)$ on the height, which has not been considered above. If the function $f(x,y,h)$ is harmonic, it must obey Laplace's equation

$$\frac{\partial^2f}{\partial x^2}+\frac{\partial^2f}{\partial y^2}+\frac{\partial^2f}{\partial h^2}=0\qquad\text{(A5)}$$

Using the techniques of separation of variables

$$f(x,y,h)=A(x)B(y)C(h)\qquad\text{(A6)}$$

$$\frac{1}{A}\frac{\partial^2A}{\partial x^2}+\frac{1}{B}\frac{\partial^2B}{\partial y^2}+\frac{1}{C}\frac{\partial^2C}{\partial h^2}=0\qquad\text{(A7)}$$

The solutions for A, B, and C are

$$A(x)=e^{ik_1x}\qquad\text{(A8)}$$

$$B(y)=e^{ik_2y}\qquad\text{(A9)}$$

$$C(h)=e^{\pm kh}\qquad\text{(A10)}$$

where

$$k=(k_1^2+k_2^2)^{1/2}$$

The Fourier series solution for $f(x,y,h)$ can be written as

$$f(x,y,h)=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\hat f(k)e^{i(k_1x+k_2y)}e^{-kh}$$

Equivalently, the Fourier transform solution is

$$f(x,h)=\frac{1}{2\pi}\int dk\hat f(k)e^{ik\cdot x}e^{-kh}\qquad\text{(A11)}$$

where x and k are two-dimensional vectors.

Equation (A11) shows that height has a very simple effect in the transform domain: it multiplies the transform by e^{-kh}. The PSD at height h, $S(k,h)$, will therefore be multiplied by e^{-2kh} since it is the square of the transform

$$S(k,h)=S(k,0)e^{-2kh}\qquad\text{(A12)}$$

The ACF at height h is then simply

$$R(\tau,h)=\int dk\ S(k,h)e^{ik\cdot\tau}$$

$$=\int dk\ S(k,0)e^{-2kh}e^{ik\cdot\tau}\qquad\text{(A13)}$$

Expressing $S(k,0)$ as the transform of $R(\tau,0)$ and substituting in Eq. (A13) produces

$$S(k,0)=\frac{1}{(2\pi)^2}\int d\tau'R(\tau',0)e^{-ik\cdot\tau'}$$

$$R(\tau,h)=\frac{1}{(2\pi)^2}\int dk\int d\tau'e^{ik\cdot(\tau-\tau')}e^{-2kh}R(\tau',0)\qquad\text{(A14)}$$

The integral over k can be done analytically by transforming to polar coordinates

$$I=\frac{1}{(2\pi)^2}\int dke^{ik\cdot(\tau-\tau')}e^{-2kh}$$

$$=\int_0^{2\pi}\frac{d\theta}{(2\pi)^2}\int_0^{\infty}k\exp[-k(2h-i\Delta\tau\cos\theta)]dk\qquad\text{(A15)}$$

where $dk = k\,dk\,d\theta$, $\Delta\tau = |\tau - \tau'|$, and $k \cdot (\tau - \tau') = k\,\Delta\tau \cos\theta$.

The integral over k is straightforward (Ref. 9, p. 92), giving

$$I = \int_0^{2\pi} \frac{d\theta}{(2\pi)^2} \frac{1}{(2h - i\Delta\tau\cos\theta)^2}$$

$$= -\frac{1}{2}\frac{\partial}{\partial h}\int_0^{2\pi} \frac{d\theta}{(2\pi)^2} \frac{1}{2h - i\Delta\tau\cos\theta}$$

$$= -2\frac{\partial}{\partial h}\int_0^{\pi/2} \frac{d\theta}{(2\pi)^2} \frac{2h + i\Delta\tau\cos\theta}{4h^2 + \Delta\tau^2\cos^2\theta} \tag{A16}$$

The imaginary part of I is zero and the real part is given in Ref. 9 (p. 152) so that

$$I = -\frac{4}{(2\pi)^2}\frac{\partial}{\partial h}\frac{\pi}{4(4h^2+\Delta\tau^2)^{1/2}} = \frac{h}{\pi(4h^2+\Delta\tau^2)^{3/2}} \tag{A17}$$

Thus

$$R(\tau,h) = \frac{h}{\pi}\int d\tau' \frac{R(\tau',0)}{[\,|\tau-\tau'|^2 + 4h^2]^{3/2}} \tag{A18}$$

Equation (A18) is the desired result – the upward continuation of the ACF of a harmonic function given on an infinite plane. The integral is identical to the upward continuation integral for a harmonic function (Ref. 6, p. 239) except that the height, h, is replaced by $2h$. If the ACF $R(\tau,0)$ is isotropic so that R depends only on the magnitude of $\tau(|\tau| = \tau)$, the angular integral in Eq. (A18) can be done analytically as follows. Let

$$d\tau' = \tau' d\theta \tag{A19}$$

$$(\tau - \tau')^2 = \tau^2 + \tau'^2 - 2\tau\tau'\cos\theta \tag{A20}$$

and the integral becomes

$$R(\tau,h) = \int_0^\infty \frac{2h\tau' d\tau' R(\tau',0)}{\pi} \times \int_0^\pi \frac{d\theta}{(4h^2+\tau^2+\tau'^2-2\tau\tau'\cos\theta)^{3/2}} \tag{A21}$$

The θ integral is given in Ref. 9 (p. 156) and yields the expression

$$R(\tau,h) = \frac{4h}{\pi}\int_0^\infty \frac{R(\tau',0)E(P)\tau' d\tau'}{[(\tau-\tau')^2+4h^2][(\tau+\tau')^2+4h^2]^{1/2}} \tag{A22}$$

where E is the complete elliptic integral of the second kind

$$E(P) = \int_0^{\pi/2} (1 - P^2\sin^2 x)^{1/2} dx \tag{A23}$$

and

$$P^2 = \frac{4\tau\tau'}{(\tau+\tau')^2 + 4h^2} \tag{A24}$$

Note that the final expression in Eq. (A22) no longer depends on the direction of τ, but only on its magnitude.

Appendix B. Third-Order Damping in the Navigation Equations

The error equations including third-order damping can generally be written as

$$\delta\ddot{r} + 2\Omega \times \delta\dot{r} + \omega_s^2 \delta r + \delta f = \delta g + \delta A \tag{B1}$$

where

$$\delta f = 2\zeta\omega_s(\delta\dot{r} - \delta V_r) - K_2\int_0^t \delta f\, dt \tag{B2}$$

and ω_s is the Schuler frequency, K_2 is the third-order damping constant, δV_r is the bias in reference velocity, δA is the error in sensed acceleration, δg is the error in the gravity model. The reason for having third-order damping is to null out the effect of a bias in the reference velocity. The Fourier transform of δf is easily computed from Eq. (B2) as

$$\hat{\delta f} = \frac{-2\zeta\omega_s\omega^2\hat{\delta r} - 2i\zeta\omega_s\omega\delta V_r}{i\omega + K_2} \tag{B3}$$

If the transform of Eq. (B1) is taken and north and east deflection components are written explicitly, the result is

$$\Delta\hat{\delta r}_E - 2i\Omega_Z\omega(i\omega+K_2)\hat{\delta r}_N = -g(i\omega+K_2)\hat{\delta\eta} \tag{B4}$$

$$\Delta\hat{\delta r}_N + 2i\Omega_Z\omega(i\omega+K_2)\hat{\delta r}_E = -g(i\omega+K_2)\hat{\delta\xi} \tag{B5}$$

where Ω_Z is the vertical component of Earth rate and

$$\Delta = i\omega^3 - \omega^2(2\zeta\omega_s + K_2) + \omega_s^2(i\omega + K_2) \tag{B6}$$

These equations are for the case where initial position, velocity, and alignment errors are zero and have assumed that only the deflection error terms are present. Equations (B4) and (B5) may be solved for $\hat{\delta r}_E$ and $\hat{\delta r}_N$, giving the transfer functions used to compute numerical results in the text.

$$\hat{\delta r}_E = \frac{\Delta(i\omega+K_2)g\hat{\delta\eta} + 2i\Omega_Z\,\omega(i\omega+K_2)^2 g\hat{\delta\xi}}{4\Omega_Z^2\omega^2(i\omega+K_2)^2 - \Delta^2} \tag{B7}$$

$$\hat{\delta r}_N = \frac{\Delta(i\omega+K_2)g\hat{\delta\xi} - 2i\Omega_Z\omega(i\omega+K_2)^2 g\hat{\delta\eta}}{4\Omega_Z^2\omega^2(i\omega+K_2)^2 - \Delta^2} \tag{B8}$$

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